L24 March 12 Connect

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It is easier and more intuitive to start with disconnected.

A space $(X, \mathcal{J})$ is disconnected if

$$\exists \underbrace{U, V \in \mathcal{J}}_{\text{particularly important}}, \ U, V \neq \phi, \ \underbrace{U \cap V = \phi, \ U \cup V = X}_{U = X \setminus V, \ V = X \setminus U}.$$

$$\therefore \left.\begin{matrix} V, X \setminus V \\ X \setminus U, U \end{matrix}\right\} \in \mathcal{J}$$

X is disconnected $\iff \exists \ \phi \neq U, V \subsetneq X$ such that U, V are both open and closed.

Qu Write the negation of disconnected

There may be several ways to write it.

* If $\phi \neq U, V \subsetneq X$ then $U \notin \mathcal{J}$ or $V \notin \mathcal{J}$ or $X \setminus U \notin \mathcal{J}$ or $X \setminus V \notin \mathcal{J}$

* If $\phi \neq U, V \subsetneq X$ and $U, V \in \mathcal{J}$ then $X \setminus U$ or $X \setminus V \notin \mathcal{J}$

Definition (useful in doing proof)

$(X, \mathcal{J})$ is connected if $\forall \ U \subset X$ that is both open and closed in X, $U = \phi$ or $U = X$.

Note that no need to mention V in above

Example $X = Y \cup G \subset \mathbb{R}^2$ where

$$Y = \{(0,y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$$

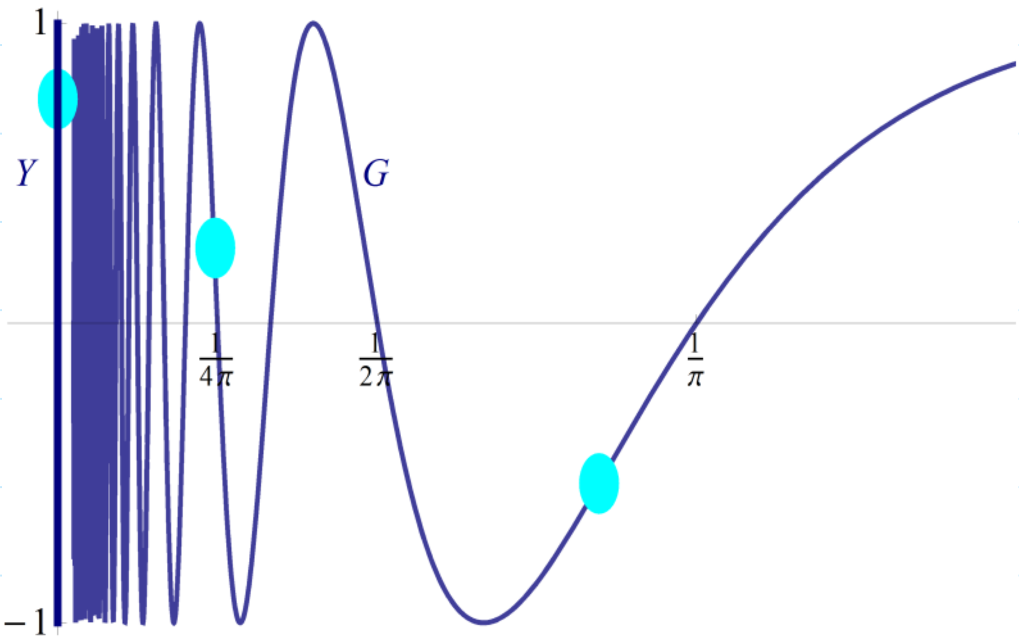
$$G = \{(x, \sin\tfrac{1}{x}) \in \mathbb{R}^2 : x > 0\}$$

X is a typical example of connected space.

Qu. How do we show it?

Let $U \subset X$ be both open and closed

Try to prove that $U = \phi$ or $U = X$



For simplicity, we accept that Y and G are connected. Consider $U \cap Y$ and $U \cap G$.

They are both open & closed in Y and G

$\therefore$ $U \cap Y = Y$ and $U \cap G = \phi$

or $U \cap Y = \phi$ and $U \cap G = G$

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For the case $U \cap Y = Y$ and $U \cap G = \emptyset$

$\exists$ open set $W \in \mathcal{T}_{\mathbb{R}^2}$ such that

$$W \cap Y = Y \quad \text{and} \quad W \cap G = \emptyset$$

In particular, $W \supset \{0\} \times [-1,1]$ and $W \cap G = \emptyset$

Using compactness of $[-1,1]$, $\exists\ \delta > 0$

$$\{0\} \times [-1,1] \subset (-\delta,\delta) \times [-1,1] \subset W$$

However $(-\delta,\delta) \times [-1,1] \cap G \neq \emptyset$

For the case of $U \cap Y = \emptyset$ and $U \cap G = G$

We may use $(X \setminus U) \cap Y = Y$ and $(X \setminus U) \cap G = \emptyset$

and the above argument.

Or, we may take a sequence $(\frac{\pi}{n}, 0) \in U \cap G \subset U$

The sequence $(\frac{\pi}{n}, 0) \longrightarrow (0,0) \notin G = U \cap G$

Thus, $U \cap G$ is not closed in G

$$\begin{array}{c} U \cap G \\ \shortparallel \\ U \cap X \end{array} \text{ is not closed in } X$$

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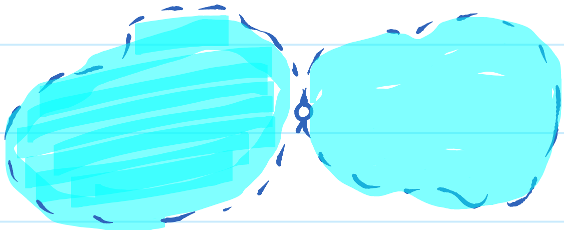
Qu. Draw a picture of a disconnected subset in $\mathbb{R}^2$, which is "almost connected"



From this example, if $X = A \cup B$ with $A \cap B = \emptyset$ the condition on A, B will determine whether X is connected or disconnected.

Theorem X be connected $\Leftrightarrow \forall\ A, B \neq \emptyset$ if $X = A \cup B$ with $A \cap B = \emptyset$ then $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$

Main Idea

By $X = A \cup B$ and $A \cap B = \emptyset$,

$$A = X \setminus B$$

If $A \cap \bar{B} = \emptyset$ then $A \subset X \setminus \bar{B}$

$$X \setminus B \subset X \setminus \bar{B}, \ \therefore\ B \supset \bar{B}$$

Thus, B is closed and A is open

Therefore $\bar{A} \cap B = \emptyset$ and $A \cap \bar{B} = \emptyset$

implies both A, B are open (and closed)

The above argument clearly goes backward.

Qu. What is the Intermediate Value Theorem?
And it analogue in high dimension?

Theorem If X is connected and $f: X \to Y$ is continuous then $f(X)$ is connected.

Proof Let $S \subset f(X)$ is both open and closed in Y

i.e. $S = G \cap f(X)$ $G \in \mathcal{T}_Y$
$= F \cap f(X)$ $X \setminus F \in \mathcal{T}_Y$

$$f^{-1}(S) = f^{-1}(G) = f^{-1}(F)$$

both open & closed in X

$\therefore f^{-1}(S) = \phi$ or $f^{-1}(S) = X$

i.e. Both $G, F \subset Y \setminus f(X)$ $S = \phi$
or both $G, F \supset f(X)$ $S = f(X)$

Theorem X is disconnected $\Leftrightarrow$
$\exists$ surjective continuous $f: X \longrightarrow (\{-1, 1\}, \text{discrete})$

"$\Rightarrow$" Let $\phi \neq U \subsetneq X$ be both open & closed

Then define $f(x) = \begin{cases} -1 & x \in U \\ 1 & x \notin U \end{cases}$ will do.

"$\Leftarrow$" Simply take $U = f^{-1}(-1)$, $V = f^{-1}(1)$, $U \cup V = X$
$U, V \neq \phi$ because f is surjective
They are open and closed because $\{-1\}, \{1\}$ are both open and closed in discrete topology.